You cannot win in a quantum casino

joint work with Cristopher Moore and Alexander Russell

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Hidden Subgroup Problem

G is a known group (usually finite);

$H \subset G$ is a hidden subgroup;

$f : G \to S$ is the oracle,
we are promised that $f(a) = f(b) \iff Ha = Hb \iff ab^{-1} \in H$;

Problem: $H = ?$

Examples

- Schor's algorithm: $G = \mathbb{Z}$,
- Simon's algorithm: $G = (\mathbb{Z}_2)^n$,
- Kuperberg's algorithm: $G = D_n$ the dihedral group,
- Graph Isomorphism Problem: $G = (S_n \times S_n) \rtimes \mathbb{Z}_2$,

Hidden Subgroup Problem, simplified

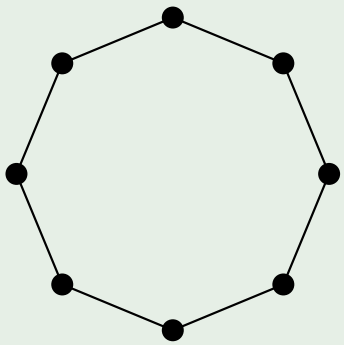
assume $N \subset G$ is a subgroup, $[G : N] = 2$,

we are promised that either

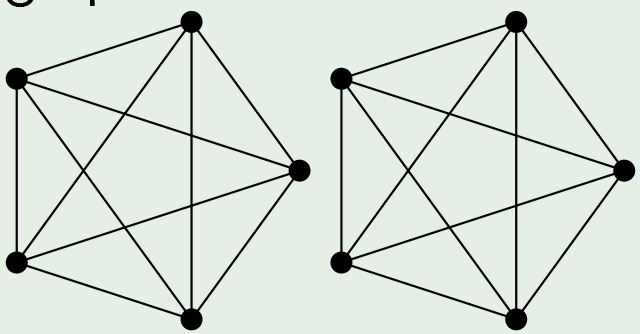
- $H = \{e\}$ is trivial, or
- $H = \{e, s\}$ with $s \notin N$

Examples

Kuperberg's algorithm:
G is the dihedral group,



Graph Isomorphism Problem:
G is the automorphism group of two copies of the complete graph



Representations

V is a (finite dimensional) vector space,
$\phi : G \to \operatorname{End} V$ is a group homomorphism:

$$\phi(ab) = \phi(a)\phi(b) \qquad \text{for any } a, b \in G;$$

we say that V is a representation of a group G

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representation V is irreducible if it cannot be written as a sum of smaller representations: $V \neq V_1 \oplus V_2$.

Example 1: discrete Fourier transform

$G = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$,
irreducible representations are indexed by $k \in \mathbb{Z}_n$,
$V_k = \mathbb{C}$ is one-dimensional,

$$\phi_k(l) = e^{\frac{2\pi i k l}{n}}$$

$$\phi_k(a+b) = \phi(a)\phi(b) \qquad \text{for any } a, b \in \mathbb{Z}_n.$$

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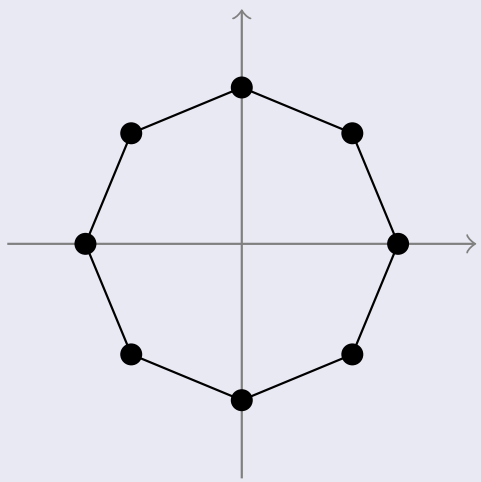
Example 2

irreducible representation of the dihedral group as symmetries of a polygon on the plane



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Example 3: the left-regular representation

the group G can be represented on the vector space

$$V = \mathcal{H} = \ell^2(G)$$

orthonormal basis: $|a\rangle_{a \in G}$

$$\phi(g)|a\rangle = |ga\rangle$$

Characters

if V is an irreducible representation of G, we consider its character

$$\chi_V(g) = \mathrm{Tr}\,\phi(g)$$

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if $N \subset G$ is a fixed subgroup and $[G : N] = 2$,
we say that an irreducible representation V of G is unlucky if

$$\chi_V(g) = 0 \qquad \text{for all } g \notin N$$

Notations

if V is a representation of G

and $H \subset G$ is a subgroup

we denote by

$$V^H = \{v \in V : \phi(h)v = v \text{ for all } h \in H\}$$

the set of H-invariant vectors

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if $W \subseteq \mathcal{H}$ is a vector space, we denote by

$$\rho_W = \frac{1}{\dim W}(\text{projection on } W)$$

the uniform mixed state on W

Coset state

Notations

we will use Hilbert space $\mathcal{H} = \ell^2(G)$ with basis $|a\rangle$, $a \in G$;
for $X \subset G$ we denote the pure state $|X\rangle = \frac{1}{\sqrt{|X|}} \sum_{a \in X} |a\rangle$

1. start with pure state $|G\rangle = \frac{1}{\sqrt{|G|}} \sum_{a \in G} |a\rangle$,
2. compute the oracle function $f : G \to S$

$$\frac{1}{\sqrt{|G|}} \sum_{a \in G} |a, f(a)\rangle \in \ell^2(G) \otimes \ell^2(S)$$

3. measure the second coordinate, forget the output

$$\rho_{H\backslash G} = \frac{1}{|G|} \sum_{a \in G} |Ha\rangle\langle Ha| = \rho_{\ell^2(G)^H}$$

the coset state

Burnside decomposition 1

$$\ell^2(G) = \bigoplus_{\substack{V \text{ - irreducible} \\ \text{representation of } G}} V \otimes V^* = \bigoplus_{V} V \oplus \cdots \oplus V$$

this decomposition can be used for a quantum measurement;

$$\rho_{G\setminus H} = \rho_{\ell^2(G)^H} = \bigoplus_{V} \rho_{V^H} \otimes \rho_{V^*}$$

the measurement gives us:

- the name of the irreducible representation V, let us hope it is lucky!
- state ρ_{V^H} (useful, but encrypted quantum information),
- state ρ_{V^*} (useless!)

Burnside decomposition 2

$$\rho_{G\setminus H} = \rho_{\ell^2(G)^H} = \bigoplus_V \rho_{V^H} \otimes \rho_{V^*}$$

Probability distribution:

- if $H = \{e\}$

$$P(V) = \frac{(\dim V)^2}{|G|}, \qquad \text{Plancherel measure}$$

- if $H = \{e, s\}$

$$P(V) = \frac{(\dim V)\big(\dim V + \chi_V(s)\big)}{|G|},$$

if V is unlucky, there is no difference
if we are unlucky we should do something more...

Quantum poker

unlucky? we can iterate this procedure: $(V_1, \rho_{V_1^H})$, $(V_2, \rho_{V_2^H})$,...

tensor product of representations:

$$V_1 \otimes V_2 = \bigoplus_V V \oplus \cdots \oplus V$$

state:

$$\rho_{V_1^H} \otimes \rho_{V_2^H} = \rho_{V_1^H \otimes V_2^H} = \bigoplus_V \rho_{(\text{some subspace of } V^H)}$$

measurement gives us:

- the name of the irreducible representation V, let us hope it is lucky!
- state $\rho_{(\text{some subspace of } V^H)}$,

and if we are unlucky...

Quantum poker, simplified

- whenever you like, you can ask the dealer for a random irreducible representation V of G,
- whenever you like, you can also...
  1. take two irreducible representations V_1, V_2 from your deck of cards, give them to the dealer,
  2. the dealer gives you back the random irreducible representation from the tensor product $V_1 \otimes V_2$,
- if you have a lucky irreducible representation, you WON!; otherwise you have to keep playing

Quantum poker

Theorem (Kuperberg)

if G is the dihedral group, there is an explicit strategy to win in a quantum poker quickly with high probability

Theorem (Moore, Russell, Śniady)

if G is the automorphism group of two copies of the complete graph, there is no strategy to win in a quantum poker quickly with high probability

proof: very strong estimates on the characters of the symmetric groups

Bibliography


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